Soliton fractals in the Korteweg–de Vries equation

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We have studied the process of creation of solitons and generation of fractal structures in the Korteweg–de Vries (KdV) equation when the relation between the nonlinearity and dispersion is abruptly changed. We observed that when this relation is changed nonadiabatically the solitary waves present in the system lose their stability and split up into ones that are stable for the set of parameters. When this process is successively repeated the trajectories of the solitary waves create a fractal treelike structure where each branch bifurcates into others. This structure is formed until the iteration where two solitary waves overlap just before the breakup. By means of a method based on the inverse scattering transformation, we have obtained analytical results that predict and control the number, amplitude, and velocity of the solitary waves that arise in the system after every change in the relation between the dispersion and the nonlinearity. This complete analytical information allows us to define a recursive *L* system which coincides with the treelike structure, governed by KdV, until the stage when the solitons start to overlap and is used to calculate the Hausdorff dimension and the multifractal properties of the set formed by the segments defined by each of the two "brothers" solitons before every breakup.

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I. INTRODUCTION

Appearance of fractal structures in soliton dynamics has been studied intensely during the past few years. The collision of solitons has revealed a path to the generation of fractal patterns. In the framework of the nonlinear Klein-Gordon equations, the velocity of two breathers after a collision revealed fractal properties associated with multiparticle effects [1]. In kink-antikink scattering processes, the transmission, reflection, or formation of a bound state depends "fractally" on the impact velocity as a consequence of the resonance between the translational and the internal modes of the kink and/or antikink [2–4]. On the other hand, in the framework of nonintegrable coupled nonlinear Schrödinger equations, the collision of vector solitons shows that the dependence of the separation velocity versus the collision velocity has a fractal structure, being explained again by a resonance mechanism between the translational motion of vector solitons and their internal oscillations [5].

Recently, a scenario where the dynamics of solitons generates fractals was proposed by Soljacic *et al.* [6]. In nonlinear-dispersive systems solitons arise as a stable configuration where the dispersive and nonlinear effects are compensated. The authors proposed that, having a soliton as an initial condition, if the relation between the parameters that govern the nonlinearity and dispersion changes nonadibatically, the soliton evolves and generates stable solitary waves for the set of parameters. If the breakup of the solitons into new ones is repetitively carried out, "soliton fractals" are induced. As an example for this path to obtain fractals, the authors ran simulations based on the nonlinear Schrödinger equation (NLS), demonstrating the generation of optical spatial soliton fractals from a single input beam [6], and showing optical temporal soliton fractal formation from a single input temporal soliton [7]. Following this idea, the manifestation of fractals in soliton dynamics has been observed experimentally [8]. The experiment utilized self-generated spin-wave envelope solitons in a magnetic film-based active feedback ring. The increase of the gain leads to a change in the relation between the dispersion and the nonlinearity [9] and generates the development of the soliton fractals.

We aim to contribute in the above described direction studying the process of the creation of solitons and generation of fractal structures in the Korteweg–de Vries equation (KdV). This well-known integrable nonlinear equation [10,11] was proposed as a model for Russell's hydrodynamical wave [12] and arises whenever one studies unidirectional propagation of long waves in a dispersive energy conserving medium at the lowest order of nonlinearity [13]. In this paper we study the dynamics of the KdV solitons when the relation between the nonlinearity and dispersion are abruptly and repetitively changed. Given a soliton as a initial condition, we use a method [14–19] based on the inverse scattering transform (IST) [10,11] to study the conditions of formation, number, and properties of the solitary waves that emerge after each nonadiabatic change in the parameter that governs the strength of the nonlinearity. This analytical information gives us the possibility to characterize and control completely the branching process that arise as a consequence of the changes in the properties of the medium where the soliton propagates. As in the case of the branching process studied by Soljacic *et al.* [6], only during a finite number of breakup processes the system shows self-similarity. After some breakup processes, the overlapping of the solitary waves starts and makes that the regular pattern of emergence of definite structures stops to emerge. The analytical results, obtained by means of the IST, allow us to bound the number

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of these iterations and describe the branching process in a recursive way as an L system [20,21], which coincides with the process described by means of KdV until the breakup where self-similarity is lost. For the case when after each breakup process each solitary wave bifurcates into two, the L system defined previously allows us to calculate analytically the Hausdorff dimension [22,23] of the set of the segments defined by each pair of "brothers" solitary waves before a new bifurcation. The Hausdorff dimension of this set is not integer and smaller than one, its value depends on the time interval between explosions and on the value of the parameter that governs the nonlinearity. The fact that the solitary waves have associated scalar magnitudes, suggest a multifractal analysis [23,24] of the evolution of the distribution of some of these magnitudes. We have associated the area of each pair of "brothers" solitary waves to the segment that they define. The multifractal analysis of this distribution of area gives a multifractal spectrum that shows the fractal properties of the distribution of the area in the system as a consequence of the bifurcation process of the solitary waves. So the structure described by the L system, that the first iterations coincides with the one described by KdV, shows self-similarity, noninteger Hausdorff dimension, and multifractal properties. So this structure is a fractal and therefore we can conclude that the process of the bifurcation of solitary waves in KdV, produced by means of the repetitive change of the strength of the nonlinearity, generates a prefractal with multifractal properties.

The structure of the paper is as follows. In Sec. II, using the inverse scattering transform formalism for KdV, we characterize analytically the solitons created after a change in the nonlinearity-dispersion relation obtaining a treelike structure. In Sec. III, we describe the branching process as an L system, that for its first iterations coincides with the branching process generated by KdV. In Sec. IV the Hausdorff dimension is obtained from the fractal described by the L system. In Sec. V we analyze the multifractal measure of the area defined by the solitary waves. Finally, Sec. VI concludes the paper by summarizing our main results and implications.

II. INSTABILITY AND FORMATION OF KDV SOLITARY WAVES SUBJECTED TO NONADIABATIC PERTURBATIONS

We have mentioned above that solitary waves in nonlinear dispersive systems lose their stability and create new ones in a branching process when the relation between the strength of the nonlinearity and the dispersion changes abruptly. In this section we will study analytically the process of branching of the KdV solitons when the parameter that control the nonlinear strength is modified.

We will study the KdV equation, using a soliton as an initial condition,

$$u_t - 6f(t)uu_x + u_{xxx} = 0,$$

$$f(t) = \begin{cases} 1, & t \leqslant 1, \\ p^n, & \sum_{j=0}^{n-1} \frac{1}{m^j} < t \leqslant \sum_{j=0}^{n} \frac{1}{m^j}, \end{cases}$$

$$u(x,0) = -\frac{2\kappa^2}{\cosh^2[\kappa(x-x_0)]}, \qquad (1)$$

where $f(t)$ is the parameter that controls the strength of the nonlinearity. So the relation between the nonlinearity and the dispersion will change nonadiabatically due to the change of $f(t)$, from $f=1$ to $f=p$ in $t=1$, $f=p^2$ in $t=1+\frac{1}{m}$, $f=p^3$ in $t=1+\frac{1}{m}+\frac{1}{m^2}$, etc. These changes cause the solitary waves present in the system to lose their stability and split up into new ones that are stable for the value of the parameter $f(t)$. We are interested in characterizing analytically this process to obtain the number and properties of the new solitons generated in each breakup.

To study the relation between the change in $f(t)$ and the number and properties of the soliton created in the bifurcation points we will use a method based on the inverse scattering transformation (IST) [10,11]. When $f(t)=1$, the solution of Eq. (1), in a class of functions decreasing when x tends to infinity, is a sum of localized and quite robust solitonlike waves and radiation [10]. Given the eigenvalues problem

$$-\varphi_{xx} + u(x,t_0)\varphi = k^2\varphi, \qquad (2)$$

the number of solitary waves in the system will come from the number of eigenvalues of the discrete spectrum $\{k_n = i\kappa_n, n=1,\ldots,N, \kappa_n > 0\}$, which control their velocity $V_n = 4\kappa_n^2$ and their amplitude $A_n = -2\kappa_n^2$ [11,14]. So, when the parameter $f(t)$ changes, it will be necessary to obtain the discrete spectrum of Eq. (2) in order to control the number and properties of solitons created in each new breakup.

In $t=1$, the parameter $f(t)$ changes from $f(t)=1$ to $f(t)=p$. So for $1<t<1+\frac{1}{m}$, $f(t)=p$ and Eq. (1) reads

$$\begin{cases} u_t - 6puu_x + u_{xxx} = 0, \\ u(x,1) = -\dfrac{2\kappa^2}{\cosh^2[\kappa(x-\phi_1)]}, \end{cases} \qquad (3)$$

where ϕ_1 is a constant. If we use the change of variables

$$y = p^{1/2}x, \quad \tau = p^{3/2}t, \qquad (4)$$

the system (3) can be written as

$$\begin{cases} u_\tau - 6uu_y + u_{yyy} = 0, \\ u(y,p^{3/2}) = -\dfrac{2\kappa^2}{\cosh^2\left[\kappa\left(\dfrac{y}{\sqrt{p}} - \phi_1\right)\right]}, \end{cases} \qquad (5)$$

and therefore it is suitable to be studied by the IST. The change of the parameter $f(t)$ makes that now $u(y,p^{3/2})$ is not a reflectionless potential [10] of the scattering problem

$$-\varphi_{yy} + u(y,p^{3/2})\varphi = k_1^2\varphi. \qquad (6)$$

We must analyze the eigenvalue problem (6) to obtain information about the number of solitonlike structures that will appear after the nonadiabatic change of the parameter $f(t)$. The eigenvalues of the discrete spectrum will come from $k_{n_1} = i\kappa_{n_1}$ with

$$\kappa_{n_1} = \frac{\kappa}{\sqrt{p}}\left(\frac{-1+\sqrt{1+8p}}{2} - n_1\right), \tag{7}$$

where n_1 is an integer such that $0 \leqslant n_1 < \frac{-1+\sqrt{1+8p}}{2}$ [25]. After the change of the value of $f(t)$ the original soliton have divided into $N+1$ "sons" solitary waves, N being the biggest integer smaller than $\frac{-1+\sqrt{1+8p}}{2}$. In the original variables these new solitary waves have the form

$$u_{n_1}(x,t) = -\frac{2\kappa_{n_1}^2}{\cosh^2[\kappa_{n_1}(\sqrt{p}x - 4\kappa_{n_1}^2 p^{3/2} t - \tilde{\phi}_0)]},$$

$$n_1 = 0,1,\ldots,N, \tag{8}$$

where $\tilde{\phi}_0$ is a constant and its velocity is given by

$$v_{n_1} = 4p\kappa_{n_1}^2 = 4\kappa^2\left(\frac{-1+\sqrt{1+8p}}{2} - n_1\right)^2,$$

$$n_1 = 0,1,\ldots,N. \tag{9}$$

These solitary waves will be stable until $f(t)$ changes its value again in $t=1+\frac{1}{m}$. In this moment each one of these $N+1$ solitons become unstable and discomposes in stable solitons for the new set of parameters. It is known [10,15–17] that if at the instant of the breakup the $N+1$ solitons are separated and if we suppose that the radiation created in the first process of breakup has a negligible role, each solitary wave evolves independently of the presence of the radiation and of the others N solitary waves. So the number and properties of the solitons generated from the n_1-solitary wave will come from the eigenvalue problem

$$-\varphi_{\tilde{y}\tilde{y}} + u_{n_1}\left[\tilde{y}, p^3\left(1+\frac{1}{m}\right)\right]\varphi = k_1^2\varphi, \tag{10}$$

where we have made again the change of variables

$$\tilde{y} = p^{1/2}y, \quad \tilde{\tau} = p^{3/2}\tau, \tag{11}$$

and the discrete spectrum of Eq. (10) is

$$\kappa_{n_1,n_2} = \frac{\kappa_{n_1}}{\sqrt{p}}\left(\frac{-1+\sqrt{1+8p}}{2} - n_2\right),$$

$$0 \leqslant n_2 < \frac{-1+\sqrt{1+8p}}{2}. \tag{12}$$

We can observe that each one of the $N+1$ "sons" of the original soliton splits up into other $N+1$ solitary waves that in the original variables have the form

$$u_{n_1,n_2}(x,t) = -\frac{2\kappa_{n_1,n_2}^2}{\cosh^2[\kappa_{n_1,n_2}(px - 4\kappa_{n_1,n_2}^2 p^3 t - \phi_{0,n_1})]}, \tag{13}$$

where ϕ_{0,n_1} is a constant phase, and their velocity are given by

$$v_{n_1,n_2} = 4p^2\kappa_{n_1,n_2}^2 = 4\kappa_0^2\left(\frac{-1+\sqrt{1+8p}}{2} - n_1\right)^2\left(\frac{-1+\sqrt{1+8p}}{2} - n_2\right)^2. \tag{14}$$

This process will be repeated for every abrupt change of the parameter $f(t)$. So after S processes of generation the system will have $(N+1)^S$ solitons (see Fig. 1), with the form

$$u_{n_1,n_2,\ldots,n_{S-1},n_S}(x,t) = -\frac{2\kappa_{n_1,n_2,\ldots,n_{S-1},n_S}^2}{\cosh^2[\kappa_{n_1,n_2,\ldots,n_{S-1},n_S}(p^{s/2}x - 4\kappa_{n_1,n_2,\ldots,n_{S-1},n_S}^2 p^{3s/2} t - \phi_{0,n_1,n_2,\ldots,n_{S-1}})]}, \tag{15}$$

where $\phi_{0,n_1,n_2,\ldots,n_{S-1}}$ is a constant and

$$\kappa_{n_1,\ldots,n_{S-1},n_S} = \frac{\kappa_{n_1,n_2,\ldots,n_{S-1}}}{\sqrt{p}} r_{n_S} = \frac{\kappa}{p^{S/2}} r_{n_1} r_{n_2} \ldots r_{n_{S-1}} r_{n_S}, \tag{16}$$

$$r_i = \frac{-1+\sqrt{1+8p}}{2} - n_i. \tag{17}$$

The velocity of each solitary wave is given by

$$v_{n_1,n_2,\ldots,n_{S-1},n_S} = 4p^s\kappa_{n_1,n_2,\ldots,n_{S-1},n_S}^2 = 4\kappa_0^2 r_{n_1}^2 r_{n_2}^2 \ldots r_{n_{S-1}}^2 r_{n_S}^2. \tag{18}$$

As we can see the solitary waves created at the same or different "explosions" have wave form of hyperbolic secant, then we can observe self-similarity between solitons of the same and different "generations."

As in other soliton fractals [6], the number of stages of the process that show self-similarity, i.e., the number of iterations for which the apparent fractal pattern appears, is finite. The formation of the apparent fractal treelike pattern, described above, stops if at the instant of a breakup of a soliton the radiation or the other solitary waves present in the system

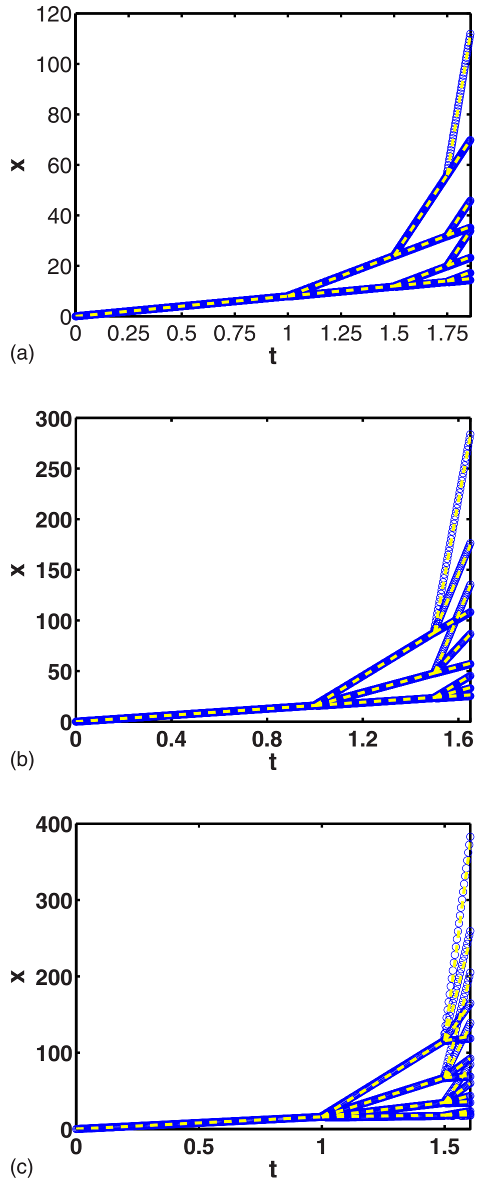

FIG. 1. (Color online) Position of the solitary waves vs time. In these panels empty circles (○) represent the position of the center of the solitons created in the branching process of the KdV equation, calculated by means of direct numerical simulations [all the simulations of the partial differential equation were carried out by means of the numerical scheme developed by Zabusky in [14] with $\Delta x=1/25$ and $\Delta t=(\Delta x)^3/4$] of the partial differential equation (PDE) (1), and yellow dashed lines correspond to the position given by the analytical results. The parameters used in the top panel are $m=2$, $\kappa=\sqrt{2}$, and $p=3$, and correspond to the case of $N=1$; each soliton divides into 2 sons. In the center panel are used $m=2$, $\kappa=2$, and $p=6$ that correspond to the case of $N=2$ and each soliton divides into 3 sons. Lastly, the bottom panel shows the case of $m=2$, $\kappa=2$, and $p=8$ that correspond to the case of $N=3$ and each soliton divides into 4 sons.

affect the properties of the solitons created in this new explosion. We can see in Fig. 1, where radiation produced in the breakups is present, that we have a very good agreement between the trajectories of the solitary waves obtained by means of numerical simulations and the trajectories predicted analytically. We can affirm that, at least for the first iterations, the radiation has a negligible effect in the process of formation of solitonlike structures and their dynamics. The fact that limits the number of iterations for which the regular pattern appears is the interaction of the solitons present in the system at the instant of a new breakup. The self-similarity and the regular treelike structure hold if the solitary waves are far enough from each other in the moment of the breakup to evolve as individual entities. As we can see in Fig. 2, if the distance between two solitons is smaller than the sum of the width of both of them during the time between breakups, they lose their shape of hyperbolic secant and the self-similarity in the same generation is lost [see Fig. 2(d)]. This implies that the discrete spectrum of the eigenvalue problem (2) is not equivalent to the case of separate solitons, so we will not obtain the same regular pattern of evolution, therefore self-similarity between different generations is lost too [see Fig. 2(f)], and the fractal treelike pattern stop to emerge.

If in the moment of the $S+1$ breakup the two slowest sons of the slowest soliton of the $S-1$ breakup are far enough from each other, any of the other solitons overlaps significantly. So the condition that guarantees that all the solitons are separated enough in the moment of the $S+1$ breakup is

$$\frac{(v_{N_1,N_2,\ldots N_{S-1},(N-1)_S}-v_{N_1,N_2,\ldots N_{S-1},N_S})}{2m^S} > \left(\frac{1}{\sqrt{v_{N_1,N_2,\ldots N_{S-1},(N-1)_S}}}+\frac{1}{\sqrt{v_{N_1,N_2,\ldots N_{S-1},N_S}}}\right). \quad (19)$$

Substituting Eq. (18) into Eq. (19) and taking into account that

$$0 < v_{N_1,N_2,\ldots N_{S-1},(N-1)_S} \leqslant 16\kappa^2,$$
$$0 < v_{N_1,N_2,\ldots,N_{S-1},N_S} \leqslant 4\kappa^2, \quad (20)$$

the following bounds holds:

$$\frac{(v_{N_1,N_2,\ldots,N_{S-1},(N-1)_S}-v_{N_1,N_2,\ldots,N_{S-1},N_S})}{2m^S} < \frac{8\kappa^2}{m^S},$$

$$\left(\frac{1}{\sqrt{v_{N_1,N_2,\ldots,N_{S-1},(N-1)_S}}}+\frac{1}{\sqrt{v_{N_1,N_2,\ldots,N_{S-1},N_S}}}\right) > \frac{3}{4\kappa}. \quad (21)$$

So if the next inequality is fulfilled

$$\frac{32\kappa^3}{3m^S} < 1, \quad (22)$$

it can be assured that the two slowest soliton overlaps and self-similarity is lost in the system.

In this section we have characterized analytically the process of branching of a KdV soliton under the change of the relation between nonlinearity and dispersion, giving rise to a self-similar pattern at least in the first iterations of the process. The complete analytical information obtained will allow us to describe recursively the process in the next section.

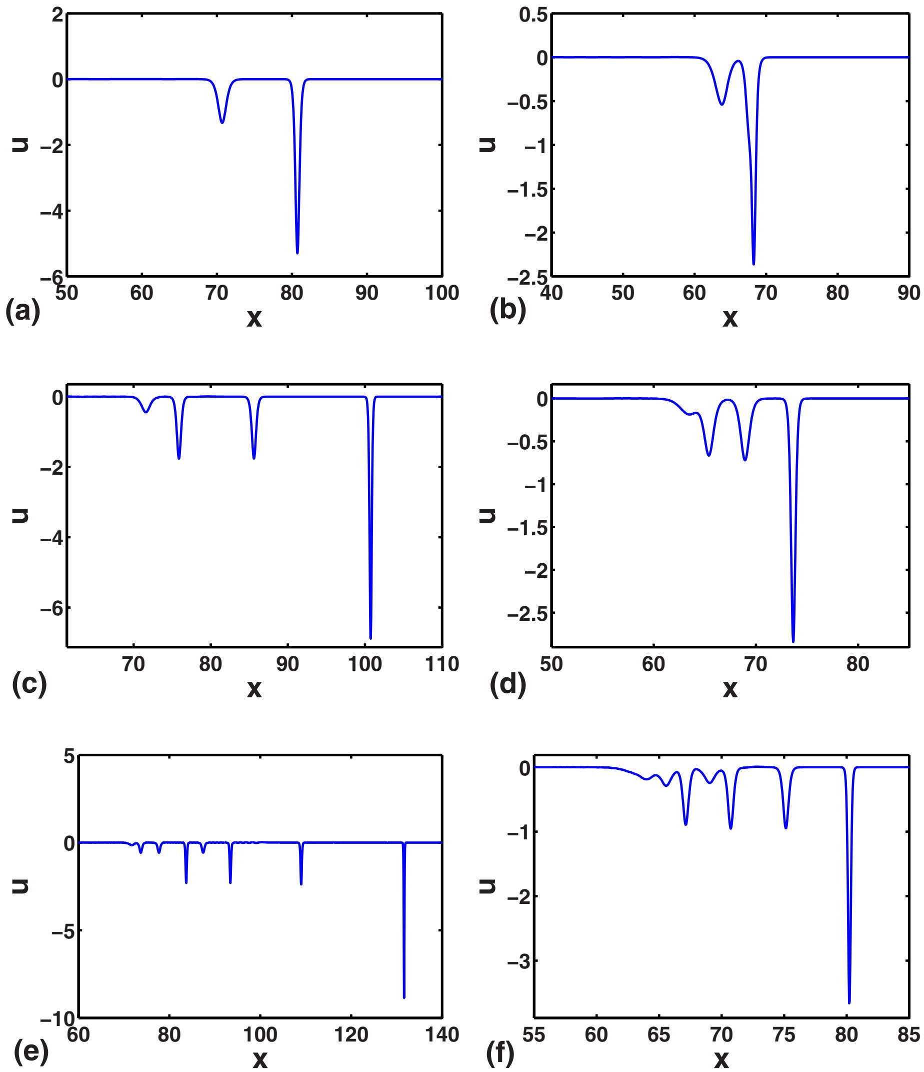

FIG. 2. (Color online) In the left-hand panels we show the branching process of the KdV equation (1), integrated numerically, for the parameters $m=2.5$, $p=3$, and $\kappa=\sqrt{2}$ that correspond to the case of $N=1$ where each soliton divides into 2 sons. Panel (a) shows $u(x)$ in $t=1.395$ just before the second breakup, panel (c) in $t=1.556$ just before the third breakup, and panel (e) in $t=1.620$ just before the fourth breakup. For all these instants of time the relation (19) is fulfilled so we can observe that for those breakups the solitons are separated enough and the regular branching pattern and the self-similarity hold. In the right-hand panels we show the branching process of the KdV equation (1) for the parameters $m=3$, $p=3$, and $\kappa=0.9$ that correspond to the case of $N=1$ where each soliton divides into 2 sons. Panel (b) shows $u(x)$ in $t=1.332$ just before the second breakup, panel (d) in $t=1.444$ just before the third breakup, and panel (f) in $t=1.481$ just before the fourth breakup. For panel (b), the relation (19) is fulfilled, so we can observe that the solitons are separated enough. Before the third breakup [panel (d)] the relation (22) is fulfilled, so the two slowest soliton overlap and the self-similarity is broken. This fact makes that for the next iterations the regular branching pattern stops to emerge, as we can see in panel (f).

III. *L* SYSTEM: RECURSIVE DEFINITION OF THE BRANCHING PROCESS

We have seen in the previous section how a soliton of the KdV equation bifurcates into a self-similar branching structure as a consequence of the process of changing the parameter $f(t)$ that governs the strength of the nonlinear term in Eq. (1). Therefore the branching process is due to the lost of stability of the solitary waves subjected to the nonadiabatic change in the relation between dispersion and nonlinearity.

The treelike structure generated in the bifurcation process (see Fig. 1) is suitable to be described by means of an *L* system language [20,21]. The main feature of the *L* system is to rewrite the process of growth of the object considered by means of a recursive process which replace in every iteration each part of the object using a set of production rules. This way of describing a geometrical object is especially useful for programming interests and it can show essential symmetries that help us to obtain a better understanding of the system under study.

To define the *L* system that describes the branching process of the KdV equation (1), we will use the so-called turtle interpretation [26]. The system will be characterized by a

TABLE I. Operators of evolution, where i is the number of noclosed brackets.

Operator	Action
F	Change the state to $\left(x+\frac{\alpha}{m^i},t+\frac{1}{m^i},\alpha\right)$ and draw a line from the old to the new state
$+$	Change the state to $(x,t,r_1^2\alpha)$
$\tilde{\mp}$	Change the state to $(x,t,r_2^2\alpha)$
[	Branch point
]	Close the branch bracket and return to the state previous to the branching

state (x,t,α) and a set of operators that act on this state. In our case each point in the plane (x,t) represents the space-time position of the center of a solitary wave and α is its velocity. Equations (16) and (18) for the number of solitons and their velocities, respectively, are used to define a set of operators describing the evolution of the solitary waves. These operators describe only the case when each soliton produces the other two, being immediate the generalization of the study of the processes of branching where each soliton generates $N+1$ sons.

In the formalism of the L systems, as in the case of the classical fractals [27], the growth of the structure will be given by the initiator, or initial cell of the system, and the generator that fixes the rule of evolution of the system for each iteration. In our case the initiator will be given by the initial soliton for $f(t)=1$ and its propagation before the first breakup

$$(0,0,4\kappa^2), \quad F, \tag{23}$$

and the generator makes that each solitary wave divides into two in every breakup and then it propagates with the velocity given by the formula (18),

$$F \to F[+F][\tilde{\mp}F],$$

$$+ \to +,$$

$$\tilde{\mp} \to \tilde{\mp},$$

$$[\to [,$$

$$] \to]. \tag{24}$$

We can observe how the L system, characterized by the rules defined in the Table I, the initiator (23), and the generator (24), describes the branching process carried out in the KdV equation when the relation between the nonlinearity and the dispersion is changed (see Fig. 3). This agreement will be so while the solitary waves are far enough from each other in the moment of the breakup to evolve as individual entities, that is while condition (19) is fulfilled. So the branching structure generated in the process governed by the KdV equation and the change of the parameter that control the strength of the nonlinearity (1) can be considered as the

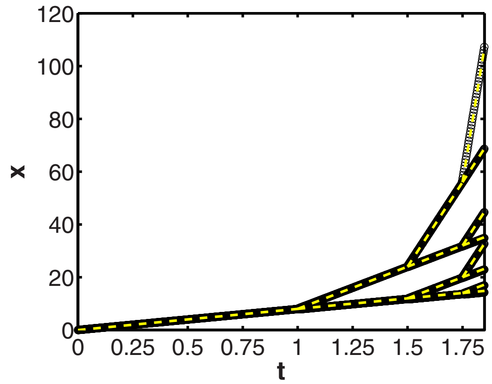

FIG. 3. (Color online) This panel shows the position of the center of the solitons generated by the branching process of the KdV equation. Results obtained by means of numerical simulation of the PDE (1) are represented by empty circles (○) and the result obtained by L system [Eqs. (23) and (24)] is represented by a yellow dashed line. The parameters used are $m=2, \kappa=\sqrt{2}$, and $p=3$.

first iterations of the branching structure generated by an L system. This fact focuses our attention on the self-similarity of its growth. The generator (24) shows that the evolution rules of each branch (solitary wave) are independent of the iteration where that branch (solitary wave) has been created.

IV. HAUSDORFF DIMENSION

The main tool to characterize a fractal is its dimension. In this section we will calculate the Hausdorff dimension [22,23] of the set, generated by the L system, that is composed by the segments that connect each pair of "brother" solitons in the moment of a new breakup (see Fig. 4). It must be pointed out that the branching structure generated by KdV (1) only coincides with the first iterations of the structure described by the L system.

For the sake of simplicity we are going to analyze the case where the parameter m is such that the branches of the pattern generated by the L system do not intersect. This condition is fulfilled if

$$\frac{r_1^2}{m-(r_1-1)^2} > \frac{(r_1-1)^2}{m-r_1^2}. \tag{25}$$

Given a "father" with velocity (18)

$$v_{Q,(N-1-Q)} = 4\kappa^2 r_1^{2Q} r_2^{2(N-1-Q)}, \tag{26}$$

after a time interval of $\frac{1}{m^N}$ the distance between its two sons will be

$$L_{Q,(N-1-Q)} = \frac{4\kappa^2}{m^N}(r_1^2 - r_2^2) r_1^{2Q} r_2^{2(N-1-Q)}. \tag{27}$$

The sum of the distance between every pair of brothers generated in the N iteration is

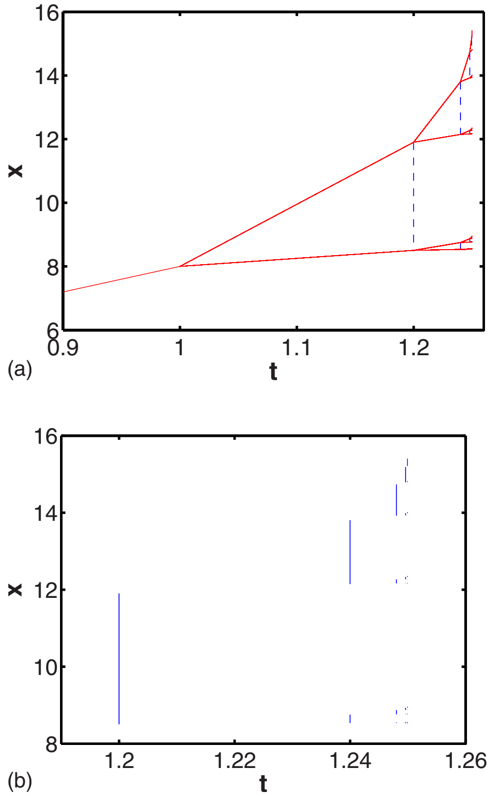

FIG. 4. (Color online) The top panel shows the branching structure defined by the L system represented by red solid lines and the segments defined for each pair of "brothers" solitons in the moment of a new breakup represented by dashed blue lines. The bottom panel shows only this set of segments. The parameters that define the L system are $m=5$, $\kappa=\sqrt{2}$, and $p=2$.

$$L=\frac{4\kappa^2}{m^N}(r_1^2-r_2^2)\sum_{Q=0}^{N-1}\binom{N-1}{Q}r_1^{2Q}r_2^{2(N-1-Q)}$$
$$=\frac{4\kappa^2}{m^N}(r_1^2-r_2^2)(r_1^2+r_2^2)^{N-1}. \tag{28}$$

If we define the limit

$$H=\lim_{N\to\infty}\left(\frac{4\kappa^2}{m^N}(r_1^2-r_2^2)\right)^d\sum_{Q=0}^{N-1}\binom{N-1}{Q}r_1^{2dQ}r_2^{2d(N-1-Q)}$$
$$=\left(\frac{4\kappa^2}{m^N}(r_1^2-r_2^2)\right)^d(r_1^{2d}+r_2^{2d})^{N-1}, \tag{29}$$

the Hausdorff dimension can be calculated as the value of the exponent d such that the limit (30) differs from infinity or zero. If we write

$$\frac{1}{m}=(r_1^{2d}+r_2^{2d})^{\alpha}, \tag{30}$$

the limit (29) is written as

$$H=\lim_{N\to\infty}[4\kappa^2(r_1^2-r_2^2)]^d(r_1^{2d}+r_2^{2d})^{N(\alpha d+1)-1}, \tag{31}$$

and it differs from zero or infinity only when

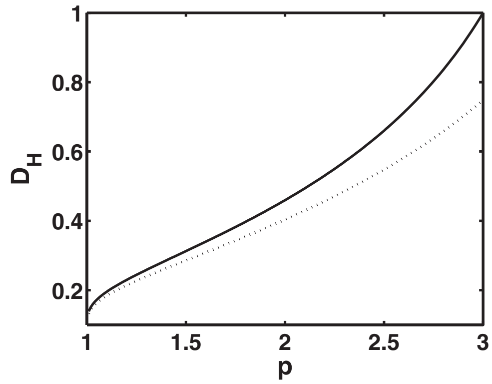

FIG. 5. The dependence of the Hausdorff dimension (D_H) of the set formed by the segments defined by each of the two "brother" solitons before every new breakup with the parameter p. The continuous line corresponds to the case of $m=5$ and the point line to $m=6$.

$$d=-\frac{1}{\alpha}. \tag{32}$$

So, the Hausdorff dimension is given by the nonlinear equation

$$d=\frac{\ln(r_1^{2d}+r_2^{2d})}{\ln(m)}. \tag{33}$$

This equation can be solved numerically (see Fig. 5) allowing us to relate the value of the Hausdorff dimension and some geometrical properties of the object. As we can see, the value of the dimension is not an integer and depends on the value of m and p being smaller than one (see Fig. 5). This fact is directly related to the properties of connectivity of our set, because structures with Hausdorff dimension smaller than one are totally disconnected [23]. The knowledge of the analytical expression of the Hausdorff dimension reveals information about some features of the topology of the branching structure generated by the L system.

V. MULTIFRACTAL MEASURE

In previous sections we have studied the geometrical features of the branching structures generated by KdV. Concretely, by means of the analytical information of the number and properties of the solitons generated in each process of change of the parameter $f(t)$, we have been able to write it as a recursive process and study the Hausdorff dimension of the set formed by the segments defined by each of the two "brother" solitons before a new breakup. There are different scalar magnitudes associated with each soliton, in this section we are going to study the geometrical properties of the distributions of the area supported by the solitons using the multifractal formalism [23,24]. To do that we are going to distribute homogeneously in each one of the segment studied in the last section the sum of the area of the two "brother" solitons that define each segment. Given a "father" with velocity (18)

$$v_{Q,(N-1-Q)} = 4\kappa^2 r_1^{2Q} r_2^{2(N-1-Q)}, \tag{34}$$

by means of Eq. (15) we can calculate the area supported by its two "sons" and assign it to the segment that they define, obtaining

$$A_{Q,(N-1-Q)} = 4\kappa(r_1 + r_2) r_1^{Q} r_2^{(N-1-Q)}. \tag{35}$$

So the total area of the set is

$$A = 4\kappa(r_1 + r_2)\sum_{Q=0}^{N-1}\binom{N-1}{Q} r_1^{Q} r_2^{(N-1-Q)} = 4\kappa(r_1 + r_2)^N. \tag{36}$$

Following the multifractal formalism [23] we cover all the segments where the area is spread with segments of length δ and we define the function

$$S_\delta(q) = \sum \mu_i^q, \tag{37}$$

where μ_i is the normalized area that is placed in the intersection of the support where the area is distributed and the i segment of length δ.

Due to the different size of each segment defined by two "brother" solitons (27), to cover the support in the iteration $N-1$ we will use segments of length

$$\delta_{N-1} = \frac{1}{B}\frac{4\kappa^2}{m^N}(r_1^2 - r_2^2) r_2^{2(N-1)}, \quad B \gg 1, \tag{38}$$

that is, their length is proportional to the size of the smallest segment of the set under study. So for a segment of length and area given by Eqs. (27) and (35), the following relations are fulfilled:

$$\left(\frac{A_{Q,(N-1-Q)}}{A}\right)^q \left(\frac{\delta_{N-1}}{L_{Q,(N-1-Q)}}\right)^{q-1}\left(1 - \frac{\delta_{N-1}}{L_{Q,(N-1-Q)}}\right) < \sum \mu_i^q,$$

$$\left(\frac{A_{Q,(N-1-Q)}}{A}\right)^q \left(\frac{\delta_{N-1}}{L_{Q,(N-1-Q)}}\right)^{q-1}\left(1 + \frac{\delta_{N-1}}{L_{Q,(N-1-Q)}}\right) > \sum \mu_i^q. \tag{39}$$

Therefore, taking into account that $\delta_N \ll L_{Q,(N-1-Q)}$, substituting Eqs. (27) and (35) in Eq. (39) and summing over all the segments we can write

$$S_{\delta_{N-1}}(q) \simeq \sum_{Q=0}^{N-1}\binom{N-1}{Q}\left(\frac{r_1^{Q} r_2^{(N-1-Q)}}{(r_1 + r_2)^{N-1}}\right)^q \times \left(\frac{4\kappa^2}{\delta_{N-1} m^N}(r_1^2 - r_2^2) r_1^{2Q} r_2^{2(N-1-Q)}\right)^{1-q}. \tag{40}$$

Thus, at the N iteration,

$$S_{\delta_{N-1}}(q) \simeq \frac{[4\kappa(r_1^2 - r_2^2)]^{1-q}(r_1^{2-q} + r_2^{2-q})^{N-1}}{(r_1 + r_2)^{q(N-1)}(m^N \delta_{N-1})^{1-q}}. \tag{41}$$

In the limit when $N \to \infty$ we can write

$$S_{\delta_{N-1}}(q) \equiv \delta_{N-1}^{-\tau(q)}, \tag{42}$$

and substituting Eq. (38) we obtain

$$\tau(q) = \frac{q \ln\left(\dfrac{r_2^2}{r_1 + r_2}\right) - \ln(r_2^2) + \ln(r_1^{2-q} + r_2^{2-q})}{\ln\left(\dfrac{m}{r_2^2}\right)}. \tag{43}$$

As we can see, $\tau(q)$ is not a lineal function [see Fig. 6(a)], so the measure of the area spread over the segments that define each of the two brother solitons shows multifractal properties [23]. By means of a Legendre transformation, we can relate $\{\tau(q), q\}$ with the new variables

$$\alpha = -\frac{d\tau}{dq}, \quad f(\alpha) = \tau(q(\alpha)) + q(\alpha)\alpha. \tag{44}$$

The curve that defines $f(\alpha)$ [see Fig. 6(b)] is known as the multifractal spectrum of the measure and reveals information about the geometrical properties of the support of the measure and about the distribution of the magnitude measured. For the case of $q=0$, it is clear that $S_{\delta_N}(0)$ corresponds to the number of δ_N segments required to cover the support, so $\tau(0)$ is the box dimension (D_B) of the set of segments defined by the brother pairs of solitons. If we invert the transformation (44) we can see that

$$q = \frac{df}{d\alpha}, \quad \tau(q) = f(\alpha(q)) - q\alpha(q), \tag{45}$$

it can be observed that for $q=0$, $f(\alpha(0))=\tau(0)$ and $df/d\alpha = 0$, that is the value of the box dimension coincides with the maximum of $f(\alpha)$ [see Fig. 6(c)]. Lastly, the value of α such that $df/d\alpha=1$ is called the information dimension (D_i). It reflects the box dimension of the support where the main part of the area is concentrated [see Fig. 6(d)].

We have shown that the branching structure generated by the process of change in the relation between the dispersion and nonlinearity in KdV and described recursively by a L System not only displays a noninteger fractal dimension (Hausdorff or box counting) but the measure of the area supported by the solitons reveals multifractal properties.

VI. CONCLUSIONS

We have studied the process of creation of solitons and generation of fractal structures in KdV equation when the parameter that controls the strength of the nonlinearity is abruptly changed. Using a one-soliton solution as the initial condition, we have observed that when this parameter is modified nonadiabatically the relation between the nonlinearity and the dispersion changes and the soliton loses its stability and generates new stable solitary waves for the new set of parameters. When the parameter that controls the nonlinearity changes the solitary waves in the system to become instable and bifurcate, this gives rise to new stable ones for the new value of the parameter. When the process is repeated, the trajectories of the solitary waves create a tree-

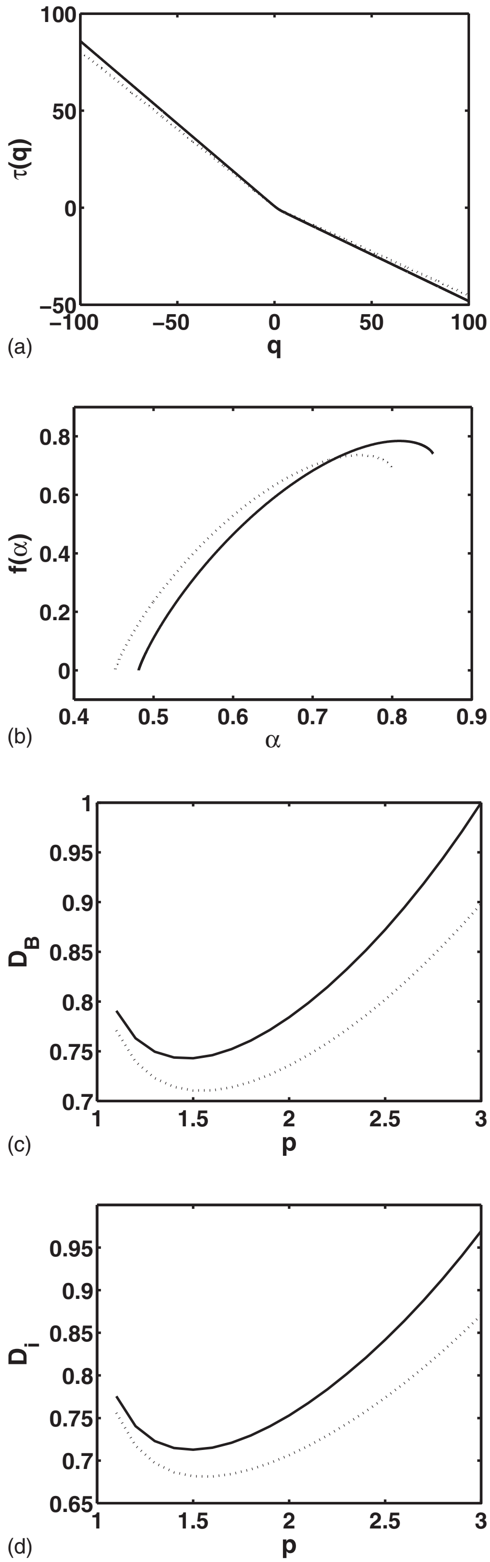

FIG. 6. (a) The dependence of the function $\tau(q)$ (43) with q for the case of $p=2$. (b) The dependence of $f(\alpha)$ with α, called multifractal spectrum, for the case of $p=2$. (c) The dependence of the box dimension (D_B) of the set formed by the segments defined by each of the two "brother" solitons before every new breakup with the parameter p. (d) The dependence of the information dimension (D_i) with the parameter p. For all the panels the continuous line corresponds to the case of $m=5$ and the dotted line to $m=6$.

like structure where successively each branch bifurcates into others.

The losing of the stability of the solitary waves and the formation of new ones has been studied by means of the IST. We have analyzed the linear problem of scattering associated to the KdV equation in all the instants where the change of the value of the parameter that controls the nonlinearity is carried out. By means of the study of the discrete spectrum associated to the scattering problem, we have characterized the solitary waves that are created in every bifurcation. The solitary waves present in the system at the same and different time instants have the form of the hyperbolic secant, so the branching structure generated due to the change in the relation between the nonlinearity and dispersion is self-similar. The self-similarity is lost when some solitary waves overlap in the moment of a new bifurcation and after that the regular branching structure stops to emerge.

The complete analytic characterization of the branching process allows us to define a recursive L system that coincides with the trajectories of the solitary waves until they start to overlap. This definition leads us to study the fractal properties of the treelike structure generated by the solitary waves. Studying the case where each solitary waves bifurcate into two and using the L system, we have analyzed the set formed by the segments defined by the brother solitary waves before a new breakup. For this Cantor-like set it is possible to determine analytically its Hausdorff dimension which is noninteger and smaller than one. On the other hand, the solitary waves carry scalar magnitudes as momentum, energy, area, etc. We have assigned the area of the brother solitary waves to the segment that they define. It allowed us to carry out a multifractal analysis of the distribution of area in the set of segments and obtain the multifractal spectrum of the measure which shows us the fractal properties of the distribution of area in the Cantor-like set.

As we have seen, the object generated by the L system is defined in a recursive way and it has self-similarity, fine structure, and noninteger Hausdorff dimension. Moreover, the study of the distribution of the area of the solitary waves shows multifractal properties. These characteristics allow us to claim that the recursive process governed by the L system generates a fractal structure. Taking into account that the first iterations of the L system coincide with the trajectories of the solitary waves governed by KdV until they start to overlap, we can affirm that the branching structure generated by the first iterations of the process described by KdV when the parameter that controls the nonlinearity is abruptly and repetitively changed is a prefractal with multifractal properties.

Finally, we would like to emphasize that the analytical description of the emergence of soliton fractals is the main contribution of this article. As we have mentioned previously, this allows complete control of the fractal structure arising in the system.

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